

1. (a) Express $1 - \sqrt{3}i$ in exponential form.
(b) Express $\ln(-5)$ in Cartesian form.
2. Use calculus to find $\int_{\pi/6}^{\pi/3} \cot 2x dx$, giving your answer in exact form.
3. Given that $\frac{dp}{dt} = t \log_e t$ and that $p = 0$ when $t = 1$, find p when $t = 2$.
4. Prove that, if $a > 0$, $\int_0^{\infty} e^{-ax} \cos bxdx = \frac{a}{a^2 + b^2}$.
5. Use Simpson's rule with 4 strips to evaluate $\int_1^5 (2x^3 - 6x) dx$.
Comment on the accuracy of this method and on whether using more strips would increase the accuracy.
6. The curve $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the x -axis. Find the volume generated.
7. A nasty disease has started to spread through an island community of 5000 susceptible people. Based on observed trends, it is predicted that the disease will spread according to the equation $\frac{dN}{dt} = 0.00002N(5000 - N)$, where N is the number of people infected and t is the time in days. On May 11, 20 people were infected. Solve the equation using partial fraction techniques and calculus. Hence find the number of people predicted to be infected by May 26. What factors might cause the actual number to differ from that predicted by the model?

- Stephanie wanted to know the diameter of her orange. So she immersed it in full beaker of water and measured the overflow. It displaced 871 cm^3 , from which she calculated the diameter to be 11.85 cm . There was a possible slight error of $\pm a \text{ cm}$ in her volume measurement. Use calculus to find an expression for the possible error in the diameter.

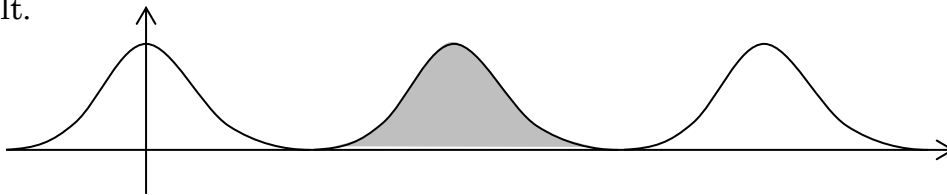
What assumptions are made in using this method for finding the diameter?

What are the strengths and limitations of this method for finding the error in the diameter?

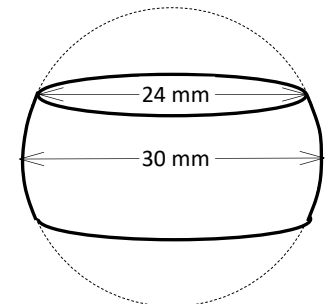
- $2 \sin 2\theta + X = 8 \sin^3\theta \cos \theta$

Replace X with a trigonometric function of θ to make the equation true for all θ .

- This graph is of $y = \cos^4 x$. Use analytical methods (not your graphics calculator) to find the shaded area. You may use your calculator to check your result.

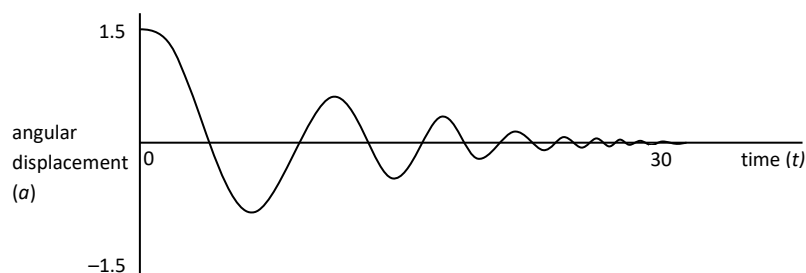


- A gold wedding ring is made in the shape of a sphere of diameter 30 mm with a cylindrical hole of diameter 24 mm cut through the middle. If the density of the gold is 19 g/cm^3 , find the mass of the ring.



Comment on the reasonableness of your result and on whether this is a suitable model for a wedding ring.

- A swinging door is opened to 1.5 radians, then let go. Its angular displacement from the closed position oscillates with decreasing amplitude **and decreasing period** as shown.

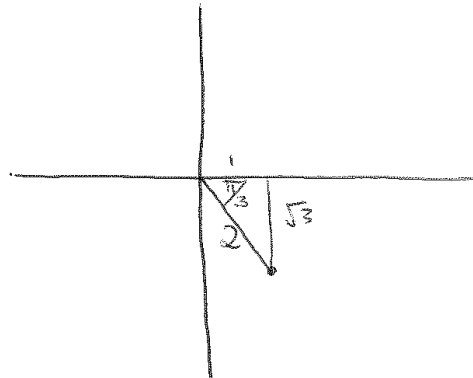


Find an algebraic formula to model the relation between angular displacement (a) and time (t) . Explain why you selected each aspect of your model. You may use your graphics calculator to help develop and refine your model function and to select appropriate values of your parameters.

Year 12 Maths C Term 2 Exam Paper A 2012

(i) (a) $1 - \sqrt{3}i$

$$= 2e^{-\frac{\pi i}{3}} \text{ or } 2e^{\frac{5\pi i}{3}}$$



(b) $\ln(-5)$

$$= \ln -1 + \ln 5$$

$$= \ln(e^{i\pi}) + \ln 5$$

$$= i\pi + \ln 5$$

$$= \ln 5 + i\pi$$

$$(2) \int_{\pi/6}^{\pi/3} \cot 2x \, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos 2x}{\sin 2x} \, dx$$

Try $\ln \sin 2x \rightarrow \frac{1}{\sin 2x} \times 2 \cos 2x$

$$\therefore \text{Integral} = \frac{1}{2} \left[\ln \sin 2x \right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2} \ln \sin \frac{2\pi}{3} - \frac{1}{2} \ln \sin \frac{\pi}{3}$$

$$= \frac{1}{2} \ln \frac{\sqrt{3}}{2} - \frac{1}{2} \ln \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0$$

$$= \ln \frac{\sqrt{3}}{2}$$

$$\textcircled{3} \quad \frac{dp}{dt} = t \ln t$$

$$\int u v' dt = uv - \int v u' dt$$

$$u = \ln t \quad v' = t$$

$$u' = \frac{1}{t} \quad v = \frac{t^2}{2}$$

$$p = \frac{t^2}{2} \ln t - \int \frac{t^2}{2} \times \frac{1}{t} dt$$

$$p = \frac{t^2}{2} \ln t - \int \frac{t}{2} dt$$

$$p = \frac{t^2}{2} \ln t - \frac{t^2}{4} + C$$

$$\text{When } t = 1 \quad p = 0$$

$$0 = \frac{1}{2} \ln 1 - \frac{1}{4} + C$$

$$C = \frac{1}{4}$$

$$\therefore p = \frac{t^2}{2} \ln t - \frac{t^2}{4} + \frac{1}{4}$$

$$\text{When } t = 2 \quad p = \frac{4}{2} \ln 2 - \frac{4}{4} + \frac{1}{4}$$

$$= 2 \ln 2 - \frac{3}{4}$$

$$\approx 0.636$$

4* RTP if $a > 0$, that $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$

LHS

$$\int_0^{\infty} e^{-ax} \cos bx \, dx$$

$$= \operatorname{Re} \int_0^{\infty} e^{-ax} e^{ibx} \, dx$$

$$= \operatorname{Re} \int_0^{\infty} e^{x(ib-a)} \, dx$$

$$= \operatorname{Re} \left[\frac{e^{x(ib-a)}}{ib-a} \right]_0^{\infty}$$

$$= \operatorname{Re} \left[\frac{e^{x(ib-a)}}{ib-a} \times \frac{ib+a}{ib+a} \right]_0^{\infty}$$

$$= \operatorname{Re} \left[\frac{e^{x(ib-a)}(ib+a)}{-b^2-a^2} \right]_0^{\infty}$$

$$= \operatorname{Re} \left[\frac{e^{-ax} e^{ibx} (ib+a)}{-b^2-a^2} \right]_0^{\infty}$$

$$= \operatorname{Re} \left[\frac{e^{-ax} (\cos bx + i \sin bx) (ib+a)}{-b^2-a^2} \right]_0^{\infty}$$

$$= \operatorname{Re} \left[\frac{e^{-ax} (ib \cos bx + a \cos bx - b \sin bx + i a \sin bx)}{-b^2-a^2} \right]_0^{\infty}$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \left[\frac{e^{-ax} (a \cos bx - b \sin bx)}{-b^2-a^2} \right]_0^{\infty}$$

$$= \left[\frac{e^{-a\infty} (a \cos b\infty - b \sin b\infty)}{-b^2-a^2} - \frac{e^0 (a \cos 0 - b \sin 0)}{-b^2-a^2} \right]$$

$$= \left[0 - \frac{a}{-b^2-a^2} \right]$$

$$= \frac{-a}{-(a^2+b^2)}$$

$$= \frac{a}{a^2+b^2}$$

= RHS

\therefore QED

3. $\frac{dp}{dt} = \frac{2t}{e^t}$
 then $\frac{dp}{dt} = 2te^{-t}$ integration by parts
 $u = 2t \quad v = -e^{-t}$
 $u' = 2 \quad v' = e^{-t}$
 then $p = -2te^{-t} + \int 2e^{-t} dt$
 $p = -2te^{-t} - 2e^{-t} + c$
 given that $p(3) = 12$ solve for c and find $p(10)$
 $12 = -6e^{-3} - 2e^{-3} + c$
 $12 = -8e^{-3} + c$
 $c = 12 + 8e^{-3} \approx 12.398$
 then $p = -2te^{-t} - 2e^{-t} + 12 + 8e^{-3}$
 $p(10) = -20e^{-10} - 2e^{-10} + 12 + 8e^{-3}$
 $p(10) \approx 12.397$

5. $\int_1^5 (2x^3 - 6x) dx$ using simpsons rule with 4 strips

$$w = \frac{5-1}{4}$$

$$w = 1$$

n	x_n	y_n
0	1	-4
1	2	4
2	3	36
3	4	104
4	5	220

$$A = \frac{w}{3} [y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$A = \frac{1}{3} [-4 + 220 + 4(4 + 104) + 2 \times 36]$$

$$A = 240$$

Simpson's method gives an exact result for cubic polynomials. So more strips will not make the result more accurate.

⑥

Slice the volume into vertical discs of thickness Δx , and x -coordinate x

For each disc

The radius = y

$$= x^2$$

The area = $\pi(x^2)^2$

The volume = $\pi x^4 \Delta x$

The total volume of all discs

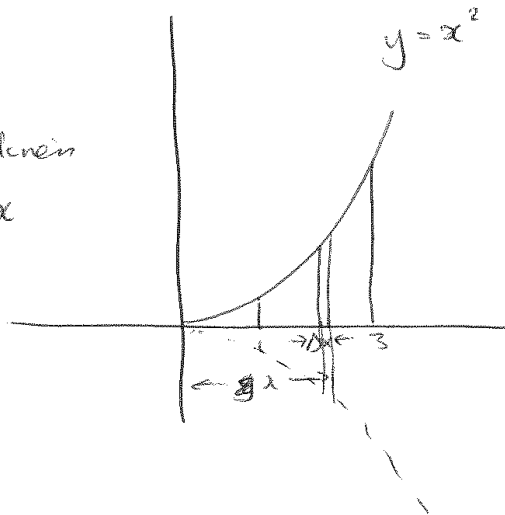
$$= \sum_{x=1}^3 \pi x^4 \Delta x$$

The exact volume = $\int_0^3 \pi x^4 dx$

$$= \pi \left[\frac{x^5}{5} \right]_0^3$$

$$= \pi \left[\frac{243}{5} - \frac{1}{5} \right]$$

$$= \frac{242\pi}{5}$$



$$(7) \quad \frac{dN}{dt} = 0.00002 N(5000 - N)$$

Let $t = 0$ on May 11, so $t = 15$ on May 26

When $t = 0$, $N = 20$

$$\int \frac{dN}{N(5000 - N)} = \int 0.00002 dt$$

$$\int \frac{1}{N(5000 - N)} dN = 0.00002 \int dt$$

$$\text{Let } \frac{1}{N(5000 - N)} = \frac{A}{N} + \frac{B}{5000 - N}$$

$$\text{So } 1 = A(5000 - N) + BN$$

$$\text{When } N = 0, 1 = A \times 5000$$

$$A = \frac{1}{5000}$$

$$\text{When } N = 5000, 1 = B \times 5000$$

$$B = \frac{1}{5000}$$

$$\therefore \frac{1}{N(5000 - N)} = \frac{1}{5000N} + \frac{1}{5000(5000 - N)}$$

$$\int \frac{dN}{N(5000 - N)} = \int \frac{dN}{5000N} + \int \frac{dN}{5000(5000 - N)} = 0.00002 \int dt$$

7 Cont'd 1

$$\int \frac{dN}{N} + \int \frac{dN}{5000-N} = 0.1 \int dt$$

$$\ln N - \ln(5000-N) = 0.1t + C$$

$$\ln \frac{N}{5000-N} = 0.1t + C$$

$$\frac{N}{5000-N} = e^{(0.1t+C)}$$

$$\frac{N}{5000-N} = e^{0.1t} e^C$$

When $t=0$ $N=20$

$$\frac{20}{4980} = e^C$$

$$\frac{N}{5000-N} = \frac{20}{4980} e^{0.1t}$$

$$4980N = 20(5000-N)e^{0.1t}$$

$$4980N = (100000 - 20N)e^{0.1t}$$

$$4980N = 100000e^{0.1t} - 20Ne^{0.1t}$$

$$N(4980 + 20e^{0.1t}) = 100000e^{0.1t}$$

7 Cont'd 2

$$N = \frac{100\,000 e^{0.1t}}{4980 + 20e^{0.1t}}$$

When $t = 15$

$$\begin{aligned} N &= \frac{100\,000 e^{1.5}}{4980 + 20e^{1.5}} \\ &= 88.4 \end{aligned}$$

So about 88 people should be infected by May 26

Factors that might cause the number to be different from this might be

- medical treatment and isolation of sufferers
- the fact that not all parts of the community are equally in contact
- the symptoms of the disease might make people more or less likely to seek social contact
- etc

Year 12 Maths C Term 2 Exam Paper B 2012

$$\begin{aligned} \textcircled{1} \quad V &= \frac{4}{3} \pi r^3 && \text{where } V \text{ is volume, } r = \text{radius} \\ &= \frac{4}{3} \pi \left(\frac{d}{2}\right)^3 && \text{where } d \text{ is diameter} \\ &= \frac{\pi}{6} d^3 \end{aligned}$$

$$\frac{dV}{dd} = \frac{\pi}{2} d^2$$

$$\frac{\Delta d}{\Delta V} \approx \frac{2}{\pi d^2}$$

$$\Delta d \approx \frac{2\Delta V}{\pi d^2}$$

$$\approx \frac{2a}{\pi d^2}$$

$$\approx \frac{2a}{11.85^2 \pi} \quad \left(\frac{2a}{441}\right) \left(\frac{a}{220.6}\right)$$

$$\approx 0.0045a$$

Assumptions: The orange is spherical

Strengths: Allows error in diameter to be found
for any small error in volume
Works well for small errors

Limitation: will become less accurate as a increases

12 Maths C Term 2 MPS Solutions

2. $2\sin 2x + X = 8\sin^3 x \cos x$

RHS

$$8\sin^3 x \cos x$$

$$= 8 \times \left(\frac{z^1 - z^{-1}}{2i}\right)^3 \left(\frac{z^1 + z^{-1}}{2}\right)$$

$$= \frac{-8}{16i} (z^3 - 3z^2z^{-1} + 3z^1z^{-2} - z^{-3})(z^1 + z^{-1})$$

$$= \frac{-1}{2i} (z^3 - 3z^1 + 3z^{-1} - z^{-3})(z^1 + z^{-1})$$

$$= \frac{-1}{2i} (z^4 + z^2 - 3z^2 - 3z^0 + 3z^0 + 3z^{-2} - z^{-2} - z^{-4})$$

$$= -\left(\frac{z^4 - z^{-4}}{2i}\right) + 2\left(\frac{z^1 - z^{-1}}{2i}\right)$$

$$= -\sin 4x + 2 \sin 2x$$

$$\therefore X = -\sin 4x$$

2. $\int \sin^4 x \, dx$

$$\sin^4 x = \left(\frac{z^1 - z^{-1}}{2i}\right)^4$$

$$= \frac{1}{16} (z^4 + 4z^3(-z^{-1})^1 + 6z^2(-z^{-1})^2 + 4z^1(-z^{-1})^3 + (-z^{-1})^4)$$

$$= \frac{1}{8} \left(\frac{z^4 + z^{-4}}{2}\right) - \frac{4}{8} \left(\frac{z^2 + z^{-2}}{2}\right) + \frac{6}{16}$$

$$= \frac{1}{8} \cos 4x - \frac{1}{2} \cos 2x + \frac{3}{8}$$

$$\therefore \int \sin^4 x \, dx = \int \frac{1}{8} \cos 4x - \frac{1}{2} \cos 2x + \frac{3}{8} \, dx$$

$$= \frac{1}{32} \sin 4x - \frac{1}{4} \sin 2x + \frac{3x}{8} + c$$

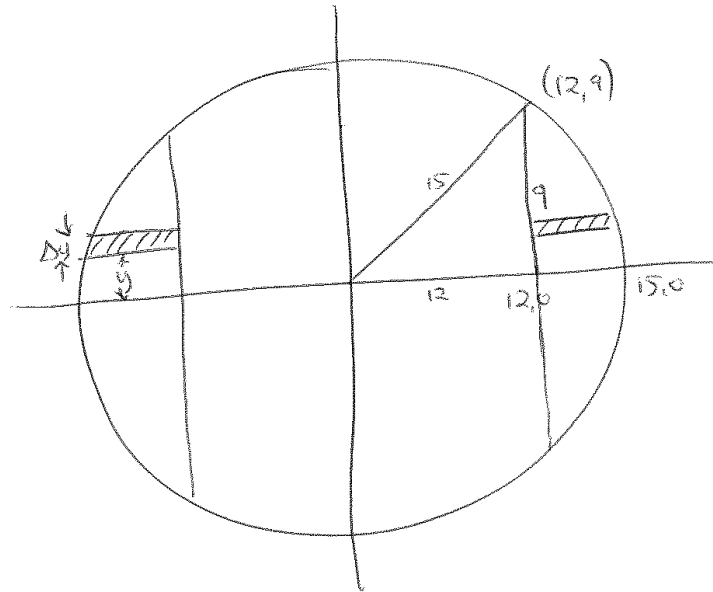
$$\begin{aligned}
 \textcircled{3} \quad \cos^4 \theta &= \left(\frac{z+z^{-1}}{2} \right)^4 \\
 &= \frac{1}{16} (z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}) \\
 &= \frac{1}{8} \left(\frac{z^4 + z^{-4}}{2} + 4 \left(\frac{z^2 + z^{-2}}{2} \right) + \frac{6}{2} \right) \\
 &= \frac{1}{8} (\cos 4\theta + 4\cos 2\theta + 3)
 \end{aligned}$$

$$\begin{aligned}
 \int \cos^4 \theta &= \frac{1}{8} \int (\cos 4\theta + 4\cos 2\theta + 3) d\theta \\
 &= \frac{1}{8} \left[\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta \right] + c
 \end{aligned}$$

The shaded area is from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$

$$\begin{aligned}
 \text{Area} &= \frac{1}{8} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^4 \theta d\theta \\
 &= \frac{1}{8} \left[\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
 &= \frac{1}{8} \left[\left(0 + 0 + \frac{9\pi}{2} \right) - \left(0 + 0 + \frac{3\pi}{2} \right) \right] \\
 &= \frac{1}{8} 3\pi \\
 &= \frac{3\pi}{8}
 \end{aligned}$$

- (4) Let the origin be at the centre of the ring and the y axis be the axis of symmetry



Let the ring be sliced horizontally as shown into slices with y-coordinate y and thickness Δy

The area of a slice is $\pi x^2 - \pi \times 12^2$

$$x^2 + y^2 = 15^2$$

$$\therefore x^2 = 15^2 - y^2$$

$$\therefore \text{Area} = \pi(15^2 - y^2) - \pi \times 12^2$$

$$= \pi(15^2 - 12^2 - y^2)$$

$$= \pi(81 - y^2)$$

The volume of a slice is $\pi(81 - y^2) \Delta y$

4 Cont'd

The sum of the volumes of all slices

$$= 2 \sum_{y=0}^9 \pi (81 - y^2) \Delta y$$

$$\text{The exact volume} = 2\pi \int_0^9 (81 - y^2) dy$$

$$= 2\pi \left[81y - \frac{y^3}{3} \right]_0^9$$

$$= 2\pi (729 - 243)$$

$$= 2\pi \times 486$$

$$= 3053 \text{ mm}^3$$

$$= 3.053 \text{ cm}^3$$

$$\begin{aligned} \text{The mass} &= 3.053 \times 19 \\ &= 58 \text{ g} \end{aligned}$$

This seems very heavy for a wedding ring.

This is because the dimensions are very large and the ring is very thick. The model will not be suitable except for a giant.

Also, no edges would be dangerously sharp.

⑤ The envelope is of the form

$$a = 1.5 e^{-kt}$$

By given and check, $k \approx 0.13$

$$\therefore a = 1.5 e^{-0.13t}$$

The wave is a cos function with b initially about 0.5, but increasing

$$\text{Try } a = 1.5 e^{-0.13t} \cos(0.5t e^{kt})$$

By given and check $k \approx 0.04$

$$\text{So } a = 1.5 e^{-0.13t} \cos(0.5t e^{0.04t})$$

$$\approx a = 1.5 \times 0.878^t \cos(0.5t \times 1.04^t)$$