Question 1 (*Modelling and Problem Solving*)

Non-polynomial functions can be written as polynomials of infinite degree.

For example, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$

This idea is used to program calculators to calculate cosines (and other functions).

Such polynomials are called power series or Maclaurin series. The power series for any differentiable function of x can be obtained using the formula:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots, \text{ i.e. } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

where $f^{(n)}(x)$ is the nth derivative of $f(x)$ and 0! is taken to be 1.

Your job is to derive this formula, i.e. to prove that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

To help you, some tips are given in the box below.

1. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
2. You may assume that the polynomial function which is the best fit to the function $f(x)$
   - has the same value as the function at $x = 0$
   - has the same first derivative as the function at $x = 0$
   - has the same second derivative as the function at $x = 0$
   - and so on
3. Using $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, find expressions for $f'(x)$, $f''(x)$, $f'''(x)$ and so on.
4. Then find expressions for $f'(0)$, $f''(0)$, $f'''(0)$ etc.
5. Then write $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ in terms of $f'(0)$, $f''(0)$, $f'''(0)$ etc.

Question 2 (*Modelling and Problem Solving*)

(a) Using the result from Question 1, derive the power series for cos x, sin x and e^x.

(b) Hence show that $e^{i\theta} = \text{cis } \theta$.

This result is known as *Euler's Formula*. It gives a further way of expressing a complex number. $a + bi$ is Cartesian form. $r\text{cis } \theta$ is a polar form known as trigonometric form. $re^{i\theta}$ is another polar form known as exponential form.

(c) Use Euler's Formula to express $4e^{2\pi/3\, i}$ in trigonometric and Cartesian form.

(d) Use Euler's Formula to express $1 + \sqrt{3}i$ in exponential form.

Question 3 (*Modelling and Problem Solving*)

(a) Euler's Formula can be expressed more generally *as* $e^{iz} = \cos z + i \sin z$ where z is any complex number. Use this fact to express $\sin z$ and $\cos z$ in terms of e^{iz} and e^{-iz}.

(b) *Euler's Identity* relates the five most fundamental numbers of mathematics, 0, 1, i, π and e. It is $e^{i\pi} + 1 = 0$. Pure mathematicians love this equation the way scientists love $E = mc^2$. The following is quoted from Wikipedia.

> A reader poll conducted by *Mathematical Intelligencer* named the identity as the most beautiful theorem in mathematics.[1] Another reader poll conducted by *Physics World* in 2004 named Euler's identity the "greatest equation ever", together with Maxwell's equations.[2]
>
> The book *Dr. Euler's Fabulous Formula* [2006], by Paul Nahin (Professor Emeritus at the University of New Hampshire), is devoted to Euler's identity; it is 400 pages long. The book states that the identity sets "the gold standard for mathematical beauty."[3]
>
> Constance Reid claimed that Euler's identity was "the most famous formula in all mathematics."[4]
>
> Gauss is reported to have commented that if this formula was not immediately apparent to a student on being told it, the student would never be a first-class mathematician.[5]
>
> After proving the identity in a lecture, Benjamin Peirce, a noted nineteenth century mathematician and Harvard professor, said, "It is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth." [6]
>
> Stanford mathematics professor Keith Devlin says, "Like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence."[7]

Use the results from Question 2 to prove Euler's Identity.

Question 4 (*Modelling and Problem Solving*)

Using the results above and showing your reasoning, express the following complex numbers in exact Cartesian form:

(a) i^{i} (b) $\ln i$ (c) $\ln(-2)$ (d) i^{e}

(e) $\cos i$ (f) $\sin(1 + i)$ (g) $\cos^{-1} i$

You may use your calculator to check your results, but not to justify them.

Questions 5 and 6 will be done under test conditions after you hand in your work on Questions 1 to 4. They are to ensure that you have understood the concepts involved in Questions 1 to 4.

Question 5 (*Knowledge and Procedures*)

Prove that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

You may assume the formula for the Maclaurin series expansion of a function.

Question 6 (*Knowledge and Procedures*)

Using the ideas from this assignment and showing your reasoning, express the following complex numbers in exact Cartesian form:

(a) i^{-i} (b) $\ln(-e)$ (c) $\cos(1 - i)$

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Q1.

Let

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

$f'^{(x)} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$

$f''^{(x)} = 2a_2 + 2 \times 3a_3x + 3 \times 4a_4x^2 + 4 \times 5a_5x^3 + \cdots$

$f'''^{(x)} = 2 \times 3a_3 + 2 \times 3 \times 4a_4x + 3 \times 4 \times 5a_5x^2 + \cdots$

$f''''(x) = 2 \times 3 \times 4a_4 + 2 \times\times 3 \times 4 \times 5a_5x + \ldots$

$f'(0) = 1 \times a_1 \qquad \therefore\ a_1 = \frac{f'(0)}{1} = \frac{f'(0)}{1!}$

$f''(0) = 1 \times 2a_2 \qquad \therefore\ a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$

$f''^{\prime(0)} = 1 \times 2 \times 3a_3 \qquad \therefore\ a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$

$f''^{\prime\prime(0)} = 1 \times 2 \times 3 \times 4a_4 \qquad \therefore\ a_4 = \frac{f''''(0)}{24} = \frac{f''''(0)}{4!}$

$\vdots$

$etc.$

$Note: f(0) = a_0 \quad \therefore\ a_0 = \frac{f(0)}{1} = \frac{f(0)}{0!}$

$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$

Q2.

a) i)

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f''''(x) = \cos x \qquad f''''(0) = 1$

$\vdots$

$etc.$

$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$\cos x = 1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

ii)

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f''''(x) = \sin x \qquad f''''(0) = 0$

$\vdots$

$etc.$

$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$\sin x = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + \cdots$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

iii)

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1$

$\vdots$

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

b) RTP $e^{i\theta} = cis\theta$

$e^{i\theta} = cos\theta + isin\theta$

sub expansion of $\cos x$ *and* $\sin x$ *into* $e^{i\theta}$

$$RHS = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$LHS = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots$$

$$= 1 + \theta i - \frac{\theta^2}{2!} + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots$$

$$= \left(1 + \theta i - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= RHS$$

$\therefore\ e^{i\theta} = cis\theta$

Q.E.D

c) $4e^{\frac{2\pi}{3}i}$

$= 4cis\left(\frac{2\pi}{3}\right)$

$= 4\cos\left(\frac{2\pi}{3}\right) + i4sin\left(\frac{2\pi}{3}\right)$

$= 4 \times \frac{-1}{2} + 4 \times \frac{\sqrt{3}}{2}i$

$= -2 + 2\sqrt{3}i$

d) $1 + \sqrt{3}i$

$= 2cos\left(\frac{\pi}{3}\right) + 2sin\left(\frac{\pi}{3}\right)i$

$= 2e^{\frac{\pi}{3}i}$

Note: need to show conversion between Cartesian and polar

Q3.

a) $e^{iz} = \cos z + i\sin z$

$e^{-iz} = \cos(-z) + \text{i}\sin(-z)$

$= \cos z - i\sin z$

$e^{iz} + \ e^{-iz} = \cos z + i\sin z + \cos z - i\sin z$

$= 2\cos z$

$\therefore \cos z = \dfrac{e^{iz} + \ e^{-iz}}{2}$

$e^{iz} - \ e^{-iz} = \ \cos z + i\sin z - \cos z + i\sin z$

$= 2i\sin z$

$\therefore \sin z = \dfrac{e^{iz} - \ e^{-iz}}{2i}$

b) RTP $e^{i\pi} + 1 = 0$

LHS

$e^{i\pi} + 1$

$= \cos\pi + i\sin\pi + 1$

$= -1 + 0 + 1$

$= 0$

$= RHS$

$\therefore Q.E.D.$

Q4.

a) i^i

$= \left(cis\ \frac{\pi}{2}\right)^i$

$= \left(e^{i\frac{\pi}{2}}\right)^i$

$= e^{i^2\frac{\pi}{2}}$

$= e^{-\frac{\pi}{2}}$

b) $\ln i$

$= \ln\left(e^{i\frac{\pi}{2}}\right)$

$= \dfrac{\pi i}{2}$

c) $\ln(-2)$

$= \ln\big(2cis(\pi)\big)$

$= \ln\big(2e^{i\pi}\big)$

$= \ln 2 + i\pi$

d) i^e

$= \left(cis\ \frac{\pi}{2}\right)^e$

$= \left(e^{i\frac{\pi}{2}}\right)^e$

$= \ e^{i\left(\frac{\pi}{2}e\right)}$

$= \cos\left(\dfrac{e\pi}{2}\right) + i\sin\left(\dfrac{e\pi}{2}\right)$

e) $\cos i = \dfrac{e^{i^2}+e^{-i^2}}{2}$

$= \dfrac{e^{-1} + e^1}{2}$

$= \dfrac{e}{2} + \dfrac{1}{2e}$

f) $sin(1+i) = sin1\,cos\,i + cos\,1\,sin\,i$

$= sin1\left(\frac{e}{2}+\frac{1}{2e}\right) + \cos 1\left(\frac{1}{2i}\left(\frac{1}{e}-e\right)\right)$ $\quad *\,Note\!: Need\ to\ show\ sini = \frac{1}{2i}\left(\frac{1}{e}-e\right)$

$= \frac{1}{2}\left(\mathrm{e}+\frac{1}{e}\right)sin1 - \frac{i}{2}\left(\frac{1}{e}-e\right)\cos 1$ $\quad Note\!: this\ simplification\ is\ not\ necessary$

$*\ \sin i = \frac{e^{i^2}-e^{-i^2}}{2i}$

$= \frac{e^{-1}-e^{1}}{2i}$

$= \frac{1}{2ei} - \frac{e}{2i}$

$= \frac{1}{2i}\left(\frac{1}{e}-e\right) \quad \times \frac{i}{i}$

$= -\frac{i}{2}\left(\frac{1}{e}-e\right)$

g) $cos^{-1}i$

$let\ z = cos^{-1}i$

$i = \cos z$

$i = \frac{e^{iz} + \ e^{-iz}}{2}$

$2i = e^{iz} + \ e^{-iz}$

$2i\left(e^{iz}\right) = \left(e^{iz}\right)^2 + \ 1$ $\qquad Note\!:\ \left(e^{-iz}\right)\left(e^{iz}\right) = 1$

$\left(e^{iz}\right)^2 - 2i\left(e^{iz}\right) + \ 1 = 0$

$e^{iz} = \frac{+2i \pm \sqrt{4i^2 - 4\times 1\times 1}}{2}$

$= \frac{2i \pm \sqrt{-8}}{2}$

$= \frac{2i \pm 2\sqrt{2}i}{2}$

$= i\ \pm\sqrt{2}i$

$= \left(1\pm\sqrt{2}\right)i$

$\therefore iz = ln\left[\left(1\pm\sqrt{2}\right)i\right]$

$z = \frac{ln\left[\left(1\pm\sqrt{2}\right)i\right]}{i} \times \frac{i}{i}$

$z = -i\ ln\left[\left(1\pm\sqrt{2}\right)i\right]$

$= -i\left[ln\left(1\pm\sqrt{2}\right) + lni\right]$ $\qquad Note\!: \ln i = ln\left(e^{\frac{\pi}{2}i}\right) = \frac{\pi}{2}i$

$= -\ln\left(1\pm\sqrt{2}\right)i - \frac{\pi}{2}i^2$

$= \frac{\pi}{2} - ln\left(1\pm\sqrt{2}\right)i$

$z = \frac{\pi}{2} - ln\left(1+\sqrt{2}\right)i \quad or\ z = \frac{\pi}{2} - ln\left(1-\sqrt{2}\right)i$

$As\ ln\left(1-\sqrt{2}\right)\ is\ imaginary - not\ in\ cartesian$

$\frac{\pi}{2} - ln\left(1-\sqrt{2}\right)i$

$= \frac{\pi}{2} - \left[\ln\left(\sqrt{2}-1\right) + \ln(-1)\right]i$

$= \frac{\pi}{2} - \left[\ln\left(\sqrt{2}-1\right) + i\pi\right]i$

$= \frac{\pi}{2} - i^2\pi - ln\left(\sqrt{2}-1\right)i$

$= \frac{\pi}{2} + \pi - \ ln\left(\sqrt{2}-1\right)i$

$= \frac{3\pi}{2} - \ ln(\sqrt{2} - 1)i$

Q5.

$$f(x) = \ln(1 + x) \qquad f(0) = 0$$
$$f'(x) = \frac{1}{1 + x} \qquad f'(0) = 1$$
$$f''(x) = \frac{-1}{(1 + x)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{2}{(1 + x)^3} \qquad f''^{\prime(0)} = 2$$
$$f''''(x) = \frac{-6}{(1 + x)^4} \qquad f''''(0) = -2 \times 3$$
$$f'''''(x) = \frac{24}{(1 + x)^5} \qquad f'''''(0) = 2 \times 3 \times 4$$

⋮

etc.

$$\therefore f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
$$\ln(1 + x) = \frac{0}{0!} + \frac{1}{1!}x - \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{2 \times 3}{4!}x^3 \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Q6.

a) i^{-i}

$$= \left(cis\ \frac{\pi}{2}\right)^{-i}$$
$$= \left(e^{i\frac{\pi}{2}}\right)^{-i}$$
$$= e^{-i^2\frac{\pi}{2}}$$
$$= e^{\frac{\pi}{2}}$$

b) $\ln(-e)$

$$= \ln e + \ln e^{i\pi}$$
$$= 1 + i\pi$$

c) $\cos i = \frac{e^{ii} + e^{-ii}}{2}$

$$= \frac{e^{-1} + e^1}{2}$$
$$\sin i = \frac{e^{ii} - e^{-ii}}{2i}$$
$$= \frac{e^{-1} - e^1}{2i}$$
$$= \frac{-i}{2}(e^{-1} - e^1)$$
$$\therefore \cos(1 - i) = \cos 1 \cos i + \sin 1 \sin i$$
$$= \cos 1 \left(\frac{e^{-1} + e^1}{2}\right) - \frac{i}{2}\sin 1\,(e^{-1} - e^1)$$