1. ROCKET – A GRAPHICAL APPROACH

In atmospheric research, rockets are often used to carry instruments into the upper atmosphere. A new rocket has been tested and it is found that, during the first 7 seconds after take-off, the height, *h*, of the rocket (in metres) is related to time, *t*, since take-off (in seconds) by the relation

$$h = 0.4\,t^3$$

Our job is to use this relation to find the velocity of the rocket at a given time.

To do this, we will start with an accurate graph of height vs time for the first 7 seconds.

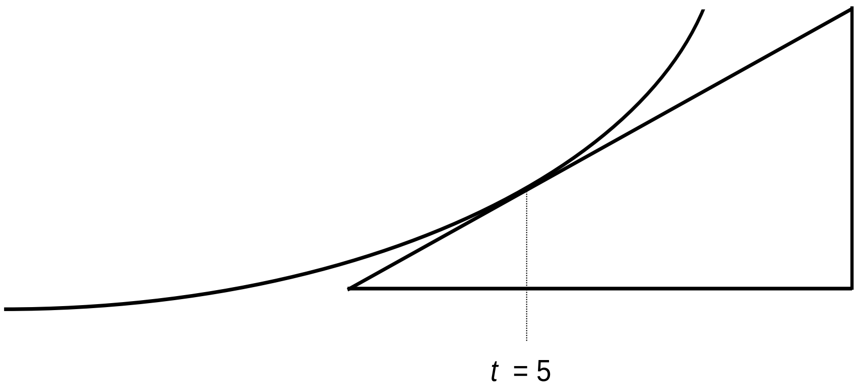
1a Produce a table for the relation above, showing values of *h* at 1-second intervals. Then use this to produce a graph of the relation.

You will notice that the graph gets steeper as you go from left to right. This is because the rocket gets faster as *t* increases.

1b At any value of *t*, the gradient of the graph is equal to the velocity at that time Present a convincing mathematical argument that this is the case.

This provides a way of finding the velocity.

1c Find the velocity at *t* = 5 by measuring the gradient of the graph at *t* = 5. Draw a tangent to the curve at *t* = 5. This tangent should have the same gradient as the curve at that point. Complete a right-angle triangle as shown below and use this to measure the gradient and hence the velocity at that time.



Don't forget that the vertical and horizontal scales on your graph will probably be different, so the gradient will have to be calculated as the change in *h* divided by the change in *t.*

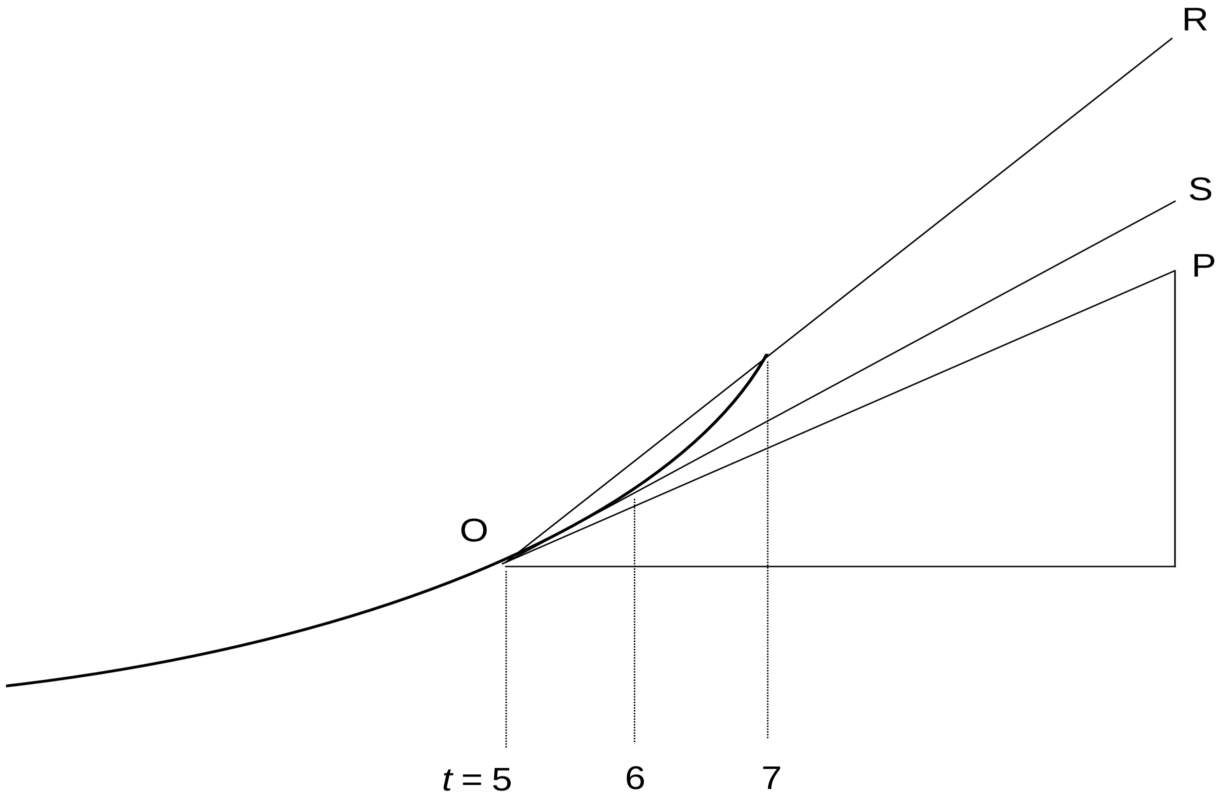
2. ROCKET – A NUMERICAL APPROACH

The graphical method above is not very accurate. An error of 5 – 10% could easily creep in through the inaccurate placement of the tangent and inaccurate measurement of the sides of the triangle. In this section, we will use a numerical approach which can give greater accuracy.

2a Calculate the average velocity of the rocket between $t = 5$ and $t = 7$ by calculating the distance travelled during this interval and dividing by the time taken (2 seconds).

The velocity at $t = 5$ will be less than the average velocity calculated above, because, by $t = 7$, the velocity has increased quite a bit over the velocity at $t = 5$.

The velocity at $t = 5$ is represented on the graph below by the gradient of the tangent OP. The average velocity between $t = 5$ and $t = 7$ is given by the gradient of the line OR



A better approximation could be obtained by calculating the average velocity from $t = 5$ to $t = 6$. This is represented by the gradient of the line OS.

2b Calculate the average velocity over the period from $t = 5$ to $t = 6$.

Using a shorter time interval still, we should get an even more accurate result. If we calculate the average velocity between $t = 5$ and $t = 5.1$, this should give us a good approximation of the velocity at $t = 5$.

If we were to magnify the part of the graph between $t = 5$ and $t = 5.1$, we would see that that part of the graph looks almost straight. The errors introduced by using a longer time period result from the change in gradient of the graph over the time interval. There is very little change in gradient between $t = 5$ and $t = 5.1$.

2c Approximate the velocity at *t* = 5 by calculating the average velocity between *t* = 5 and *t* = 5.1.

An even better approximation would be obtained by using an even shorter time interval.

2d Perform the necessary calculations and complete the following table.

Time interval used	Calculated velocity
t = 5 to *t* = 7	
t = 5 to *t* = 6	
t = 5 to *t* = 5.1	
t = 5 to *t* = 5.01	
t = 5 to *t* = 5.001	
t = 5 to *t* = 5.0001	

We should expect that the calculated velocities in the table above will get closer and closer to the true velocity at *t* = 5. In fact they seem to be getting closer and closer to a particular round number. Scientific calculators do not carry enough decimal places to be able to go any further with the table, but if we could, we might expect the results to keep on getting closer and closer to this round number. In fact they would.

We will never get the number exactly, but we must always go some small amount past *t* = 5 with our calculations. To get an exact result we would have to find the average velocity between *t* = 5 and *t* = 5. We would discover that the rocket travelled 0 metres in 0 seconds – not very helpful.

But we can see where our results are heading, and, in fact, the true velocity at *t* = 5 is the round number the results are heading towards.

2e What do you think the true velocity is at *t* = 5?

2f Calculate the velocity by using the time interval from *t* = 5 to *t* = 5 + *a*.

The resulting expression should contain 3 terms, one without *a*, one with *a* and one with a^2. If we make *a* very small, say 0.000 000 000 000 000 001, then the terms with *a* and a^2 will be insignificant compared to the term without *a*, and so we can ignore them.

2g By ignoring the terms with *a* and a^2, calculate the velocity at *t* = 5.

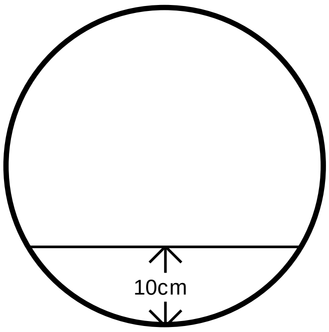
2h Calculate the velocity by using the time interval from *t* = *t* to *t* = *t* + *a*.

Leave the variable *t* in the expression rather than using 5. The formula will then give the velocity at any time: all we have to do is substitute the chosen value for *t*.

2i Use the velocity formula you found in 2h to find the velocity at *t* = 5, *t* = 2 and *t* = 7.

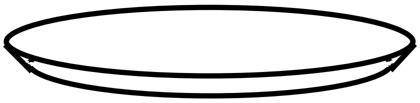
3. BUOY

A buoy is made from a hollow steel sphere. Its internal diameter is 40cm. The buoy has been floating in the water in the Logan River Estuary for several years and has leaked so that there is now water inside it. The depth of the water at the centre is 10cm.



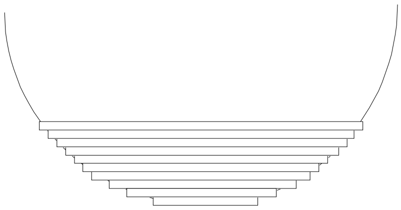
Your job is to find the volume of water in the buoy. You probably don't know a method to do this exactly, so we will use a method that will give us an approximate answer.

We will consider the water in the buoy to be made up of 10 horizontal layers, each 1cm thick.



Although the layers are not quite cylindrical, most are approximately so, so we will consider them as cylinders. We will take the radius of each layer as being the radius of the upper surface of that layer. We can then calculate the approximate volume of each layer. Obviously our calculated volume will be a bit larger than the true volume. The radius of each cylinder can be calculated using Pythagoras.

3a Calculate the approximate volume of each of the 10 layers and add them up to find the approximate volume of the water.



What we have calculated is the volume of the stack of cylinders in the diagram above. This is obviously a bit larger than the volume of the water. We would get a better approximation by considering the water as made up of 100 layers, each 0.1cm thick.

3b Explain why. A diagram will probably assist your explanation.

The problem with using 100 layers is that the calculation will take very much longer. We can get around this by using a spreadsheet.

3c Produce a spreadsheet and use it to find the volume setting the number of layers to 10, 100, and 1000. Use your results to complete the following table.

Number of slices (*n*)	**Calculated volume**
10	
100	
1000	

3d From your table give the best estimate you can of the exact volume of the water. Explain your reasoning.

Note that, unlike in the rocket example, the true volume is not a round number, but is a product of π and is therefore irrational.