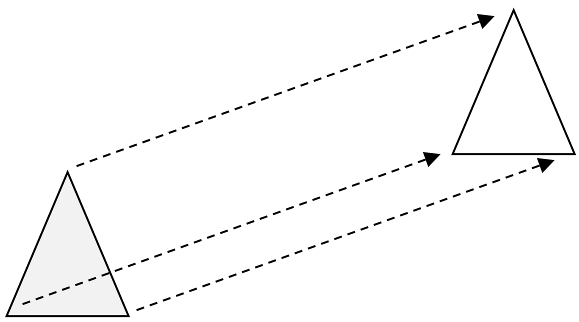
Transformations

A transformation is a change in the location of a set of points. We will look at transformations of the points on the Cartesian plane.

Translations

In a translation, all points are moved with the same displacement, e.g. 2 to the right and 3 down. Thus, in a translation, shapes keep the same size, orientation and shape. Only their position changes.



If we express points in 2 dimensions as position vectors on the Cartesian plane, we can perform a translation by adding the same given vector to each point. For instance, to move all points 2 to the right and 3 down, we add the vector (2, –3) to each point. So (0, 0) moves to (2, –3), (4, 4) moves to (6, 1) and so on.

Q1. Draw a Cartesian grid, mark on it the points P (0, 3) and Q (2, –4), then mark on their images P' and Q' after a translation of '3 to the left and 1 up'.

Q2. On another Cartesian grid, mark on it the points P (1, 3) and Q (–1, 2), then mark on their images P' and Q' after a translation of (–4, 2).

Q3. On another Cartesian grid, draw the triangle with vertices (0, 3), (2, 2) and (–1, 2), then draw its image after a translation of (4, 0).

Linear Transformations – Geometric Approach

A transformation can be thought of as a movement of each point (x, y) to a new point (x', y') according to some rule.

A transformation is linear if
- points that are in a straight line before the transformation are also in a straight line afterwards and
- a point at the origin remains at the origin.

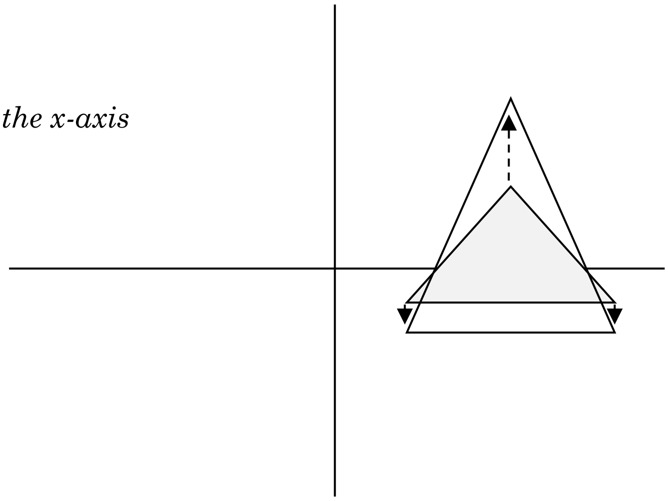
Translations aren't linear transformations because they don't satisfy the second requirement.

Geometrically speaking, there are four basic types of linear transformation – dilation, reflection, rotation and shear. Combinations of these are also linear transformations.

Dilation

In a dilation about the x-axis, the y-coordinate of each point is multiplied by some constant, the dilation factor. In a dilation about the y-axis, the x-coordinate of each point is multiplied by some constant.

Dilation of 2 about the x-axis



The effect on shapes is to stretch them or compress them vertically and/or horizontally.

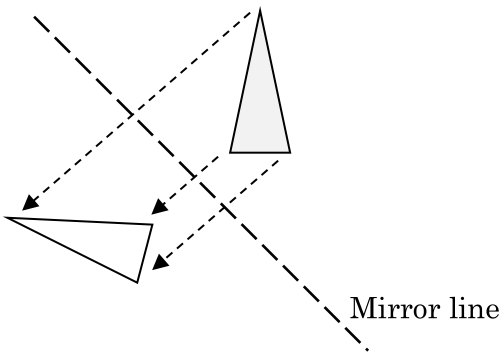
Q4. Give the new coordinates of the point (2, 1) when dilated about the x-axis using dilation factors of (a) 3 (b) 0.5 (c) –2 (d) –1

Q5. Give the new coordinates of the point (–4, 0.5) when dilated about the y-axis using dilation factors of (a) 1 (b) 2 (c) –2 (d) –0.2

Q6. Draw a square with corners at (–1, 1), (2, 1), (–1, 4) and (2, 4). Then, on the same axes, draw its image after a dilation of 2 about the y-axis, followed by a dilation of 0.5 about the x-axis.

Reflection

In a reflection each point is reflected in a given line, the mirror line, i.e. it is moved perpendicularly towards the line, then continues an equal distance past it. So, for example, if (2, 4) is reflected in the y-axis, it will move to (–2, 4).



For a reflection to be a linear transformation, the mirror line must pass through the origin. To reflect points in lines other than the axes, some geometric or trigonometric calculation may be necessary.

Q7. On a Cartesian plane, draw the line $x = 0$ and reflect the following three points in it, giving their new coordinates: (2, 3), (–5, 0), (0, –2)

Q8. On a Cartesian plane, draw the line $y = x$ and reflect the following three points in it, giving their new coordinates: (2, 3), (–5, 0), (0, –2)

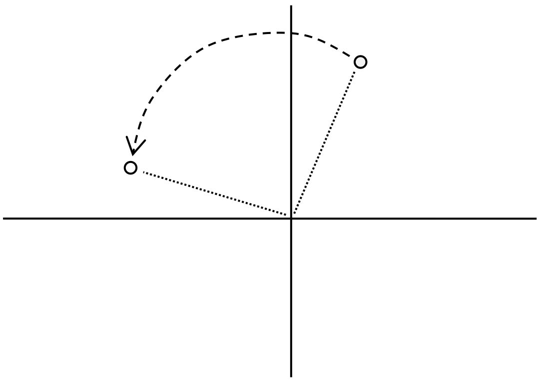
Q9. On a Cartesian plane, draw the line $y = -\sqrt{3}\,x$ and reflect the point (–5, 0) in it, giving the coordinates of the image.

Rotation

In a rotation each point remains the same distance from a given point (the centre of rotation or axis of rotation), but rotates a given angle around that point. By convention we take anticlockwise rotations as positive, clockwise rotations as negative.

For a rotation to be a linear transformation, the centre of rotation must be the origin.

Rotating the point (2, 5) 90° around the origin takes it to (–5, 2).



As with reflection, some geometric calculations may be necessary for rotations other than multiples of 90°.

Q10. Find the image of the point (2, 4) under rotations about the origin of:
(a) 90° (b) –90° (c) 180° (d) 45° (e) 115°

Shear

In a shear parallel to the x-axis every point (x, y) is moved to a new point $(x+sy, y)$, where s is the shear factor. This gives the same effect as drawing a shape on the side of a stack of cards, then pushing the top of the stack sideways.

In the same way, in a shear parallel to the y-axis every point (x, y) is moved to a new point $(x, y+sx)$.

Q11. Find the image of (2, 3) after a shear of 2 parallel to the y-axis.

Q12. Find the image of (1, 3) after a shear of –0.4 parallel to the x-axis.

Linear Transformations – Algebraic Approach

All transformations can be expressed algebraically. We can give an algebraic expression for the new coordinates (x', y') in terms of the original coordinates (x, y)

The following is one such rule:

$$x' = x^2 + 2y - 4$$
$$y' = xy + 2x - \cos x$$

This is not a linear transformation and we do not have to be concerned with it. A linear transformation has a rule of the form

$$x' = ax + by$$
$$y' = cx + dy$$

This can be expressed in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

There is a one-to-one correspondence between the set of linear transformations in 2 dimensions and the set of 2×2 matrices. In other words, every linear transformation can be written as a 2×2 matrix and every 2×2 matrix represents a linear transformation.

Finding an image given the transformation matrix

To find the image of a point, say (3, –5), under a given transformation, say $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$, we just multiply the vector for the original point on the left by the transformation matrix. So the image of (3, –5) is $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$.

Q13. Find the images of the points (2, 3), (0, 4), (3, –1) and (0, 0) under the transformation $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$.

Q14. Find the images of the points (2, 3), (0, 4), (3, –1) and (0, 0) under the transformation $\begin{pmatrix} 1 & -2 \\ 3 & -3 \end{pmatrix}$.

Finding the matrix for a given transformation

Finding the matrix for a given transformation is easy. In the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

if we let (x, y) be (1, 0), we get $(x', y') = (a, c)$;
if we let (x, y) be (0, 1), we get $(x', y') = (b, d)$.

So all we have to do is find the image of (1, 0): this gives us the numbers a and c.
Then we find the image of (0, 1): this gives us b and d.
We can find these images using the geometric methods used in Q 1-12.

For example, suppose we needed to find the matrix for a rotation of 180°. Under this rotation, the point (1,0) will move to (–1, 0), so the left column of the matrix is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. And the point (0, 1) will move to (0, –1), so the right column will be $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. So the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Q15. Find the matrix for each of the following transformations, then use the matrix to find the image of (3, –4) under that transformation.

a. a dilation of 2 about the x-axis
b. a reflection in the y-axis
c. a reflection in the line $y = x$
d. a reflection in the line $y = x \tan 30°$
e. a rotation of 90° about the origin
f. a rotation of 135° about the origin
g. a rotation of –20° about the origin
h. a shear of 1 parallel to the y-axis
i. a shear of –3 parallel to the x-axis

Successive Transformations

Two (or more) successive linear transformations can be performed on the points that make up the Cartesian plane. The result will always be a single linear transformation.

Consider reflection in the y-axis followed by a dilation of 2 about the x-axis. The net result is not a simple dilation, reflection, rotation or shear. But it is a linear transformation in that straight lines will still be straight and a point at the origin will still be at the origin.

The combined transformation will correspond to a matrix and it is easy to find that matrix.

Consider a point (x, y). The first transformation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; the second is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

The image of (x, y) after the first transformation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

The image after the second transformation is then $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

As matrix multiplication is associative, the combined transformation can be represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

Note that, because matrix multiplication is not commutative, the matrix for the second transformation has to be placed to the left of the matrix for the first transformation.

Q16. Find single matrices for the following sets of transformations:

a. $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ followed by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ followed by $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$

c. $\begin{pmatrix} -0.5 & 1 \\ -1 & 2 \end{pmatrix}$ followed by $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

d. $\begin{pmatrix} -0.5 & 1 \\ -1 & 2 \end{pmatrix}$ followed by $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ then by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Q17. Find the single matrix for each of the following sets of transformations:

a. reflection in the x-axis followed by rotation of 180° about the origin
b. dilation of 2 about the x-axis followed by a dilation of 0.5 about the y-axis, then a shear of 2 parallel to the y-axis
c. a rotation of –45° about the origin followed by reflection in the line $y = -x$.

Inverse Transformations and Singular Transformations

For most linear transformation, there is an inverse transformation which will undo it, i.e. return all the image points back to their original location. For example, the inverse of a rotation of 70° is a rotation of –70°.

The matrix for the inverse of a transformation is just the multiplicative inverse of the matrix for the original transformation. The inverse of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$, i.e. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Q18. Find the inverse of each of the following transformations:

a. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b. $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

The transformation in the last question was a dilation of 0 about the x-axis. With this transformations, the y-coordinates of all points become 0. So (4, 2) is transformed to (4, 0) and (4, 4) is transformed to (4, 0). This transformation cannot be undone because we do not know whether the point (4, 0) should go back to (4, 2) or (4, 4) or (4, anything else). That is why you found that the matrix was singular (has no inverse). Any transformation which maps two or more points onto the same image point is singular and has no inverse.

Q19. The point P is transformed using the transformation $\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$ followed by the transformation $\begin{pmatrix} 1 & 0.5 \\ -1 & 0 \end{pmatrix}$. If its final image is (4, –1), what were its original coordinates?

Revision and MAPS

Q20. Find the matrix for a reflection in the line $y = 2x$ and find the image of the point (3, 4) under that transformation.

Q21. Find the matrix for a rotation of –40°. Find the original coordinates of a point which is transformed to (4, 1.5) under this rotation.

Q22. Find a single matrix that will produce a shear of 1 parallel to the x-axis followed by a dilation of ½ about the y-axis. Where will a point at the origin end up as a result of these transformations?

Q23. Under the transformation $\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}$, the image of the point P is (0, 4). Find the original coordinates of P.

Q24. Under the transformation $\begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$, the image of the point P is (3, 2). Find the original coordinates of P. Comment on your result.

Q25. The line $y = 3x - 2$ is transformed using the transformation $\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$. What will be the equation of the image?

Answers to Transformations

Q1. P' is (–3, 4), Q' is (–1, –3)
Q2. P' is (–3, 5), Q' is (–5, 4)
Q3. The new vertices are at (4, 3), (6, 2) and (3, 2)
Q4. (a) (2, 3) (b) (2, 0.5) (c) (2, –2) (d) (2, –1)
Q5. (a) (–4, 0.5) (b) (–8, 0.5) (c) (8, 0.5) (d) (0.8, 0.5)
Q6. Corners at (–2, 0.5), (4, 0.5), (–2, 2), (4, 2)
Q7. (–2, 3), (5, 0) and (0, –2)
Q8. (3, 2), (0, –5) and (–2, 0)
Q9. $(2.5, 2.5\sqrt{3})$,
Q10. (a) (–4, 2) (b) (4, –2) (c) (–2, –4) (d) $(-\sqrt{2}, 3\sqrt{2})$ (e) (–4.433, 0.588)
Q11. (2, 7)
Q12. (–0.2, 3)
Q13. (7, 6), (4, 0), (5, 9), (0, 0)
Q14. (–4, –3), (–8, –12), (5, 12), (0, 0)
Q15. a. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (3, –8) b. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (–3, –4) c. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (–4, 3)

d. $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ $(\frac{3}{2} - 2\sqrt{3}, \frac{3\sqrt{3}}{2} + 2)$ e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (4, 3) f. $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ $(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}})$

g. $\begin{pmatrix} 0.9 & -0.34 \\ -0.34 & 0.9 \end{pmatrix}$ (4.876, 1.107) h. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (3, –1) i. $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ (15, –4)

Q16. a. $\begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix}$ b. $\begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$ c. $\begin{pmatrix} -2 & 4 \\ -3 & 6 \end{pmatrix}$ d. $\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$

Q17. a. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 0.5 & 0 \\ 1 & 2 \end{pmatrix}$ c. $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Q18. a. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ b. $\begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}$ c. $\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$ d. doesn't exist

Q19. $(^{13}/_4, ^1/_2)$

Q20. $\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$ (1.4, 4.8)

Q21. $\begin{pmatrix} 0.766 & 0.643 \\ -0.643 & 0.766 \end{pmatrix}$ (2.100, 3.721)

Q22. $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ at the origin

Q23. (–3, 1)
Q24. The matrix is singular so the coordinates of P cannot be determined.
Q25. $y = {}^4/_3x - {}^4/_3$