Knowledge and Procedures

1. The following is an encrypted message: 8448856744756870
   The following method was used to encrypt it:
   - split the message into groups of four letters, then, for each group:
     - replace each letter with the number corresponding to its position in the alphabet to get a, b, c, d
     - arrange these numbers into a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
     - find the matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix}$
     - arrange the elements of $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ as a sequence e, f, g, h
   - reassemble the message in order

   Decipher the message.

2. Find the volume of a parallelepiped if one corner is at (0, 2, 1) and edges run from that corner to (0, 3, –1), (2, 3, 0) and (1, 4, 3).

3. Use finite differences to find a formula for $\sum_{r=1}^{x} r^2$.

Modelling and Problem Solving

4. Find a matrix which will produce a single transformation of the x-y plane equivalent to the following sequence:
   - a rotation of 70° about the origin
   - a reflection in the x-axis
   - a rotation of 130° about the origin
   - a reflection in the y-axis
   - a rotation of 60° about the origin
   - a dilation by a factor of 2 from the origin

5. If the sum of the sequence 1, 3, 7, 15, 31, 63 . . . to a million terms is $2^x + y$, find a value of $x + y$.

6. Poggles is in the northern hemisphere 10 000 km by air from Brisbane (27°S 153°E) and 10 000 km by air from Singapore (1°N 104°E). Find his latitude and longitude. [Note that 10 000 km is one quarter of the circumference of the Earth.]

THE END

☺

12 Maths C Term 4 Exam Solutions 2012

(1) $\underset{\sim}{A} = \begin{pmatrix} 84 & 48 \\ 85 & 67 \end{pmatrix}$ $\underset{\sim}{B} = \begin{pmatrix} 44 & 75 \\ 68 & 70 \end{pmatrix}$

$$\underset{\sim}{A} - \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} = \begin{pmatrix} 34 & -2 \\ 35 & 17 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2+1} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 34 & -2 \\ 35 & 17 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 69 & 15 \\ 36 & 36 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 5 \\ 12 & 12 \end{pmatrix}$$

This is WELL

$$\underset{\sim}{B} - \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} = \begin{pmatrix} -6 & 25 \\ 18 & 20 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -6 & 25 \\ 18 & 20 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 12 & 45 \\ 42 & 15 \end{pmatrix}$$

$$= \begin{matrix} 4 & 15 \\ 14 & 5 \end{matrix}$$

This is DONE

So the message is WELL DONE

(2) The edge vectors are

$$(0,3,-1)-(0,2,1) = (0,1,-2)$$
$$(2,3,0)-(0,2,1) = (2,1,-1)$$
$$(1,4,3)-(0,2,1) = (1,2,2)$$

The volume is the scalar triple product, which is

$$\begin{vmatrix} 0 & 1 & -2 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= -1\times(2\times2+1\times1) - 2(2\times2-1\times1)$$
$$= -5-6$$
$$= -11$$

$\therefore$ The volume is 11

③ $y = \sum_{r=1}^{x} r^2$

x	0	1	2	3	4	5	6
y	0	1	5	14	30	55	91

1 4 9 16 25 36

3 5 7 9 11

2 2 2

Δ^3 is constant, so it is a cubic of the form

$y = ax^3 + bx^2 + cx + d$

$\Delta^3 = ad!$

$2 = a \times 6$

$a = 1/3$

When $x = 0$, $y = 0$ $\therefore$ $d = 0$

So the formula is $y = \frac{1}{3}x^3 + bx^2 + cx$

When $x = 1$ $y = 1$

$\therefore 1 = \frac{1}{3} + b + c$

$b + c = 2/3$ ... ①

When $x = 2 \quad y = 5$

$\therefore \quad 5 = \frac{1}{3} \times 8 + 4b + 2c$

$4b + 2c = 2\frac{1}{3} \quad \ldots (2)$

(1) $\times 2 \Rightarrow \quad 2b + 2c = 1\frac{1}{3} \quad \ldots (3)$

(2) $-$ (3) $\Rightarrow \quad 2b = 1$

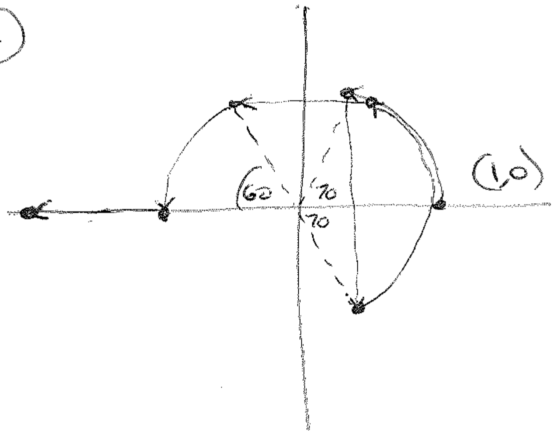
$b = \frac{1}{2} \quad \ldots (4)$

(4) into (1) $\Rightarrow \quad \frac{1}{2} + c = \frac{2}{3}$

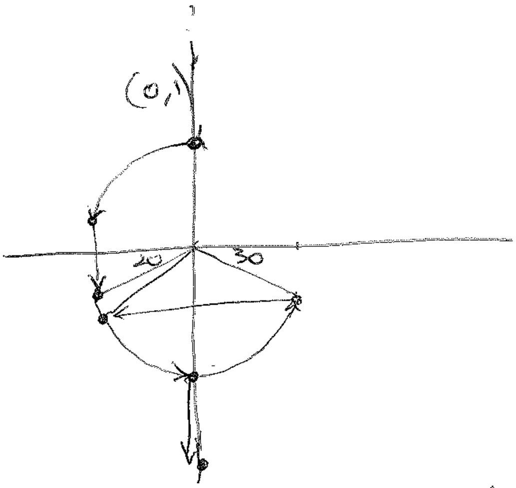
$c = \frac{1}{6}$

$\therefore$ The formula is $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$

(4)



(1,0) is transformed to (-2,0)



(0,1) is transformed to (0,-2)

So the transformation matrix is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

(5) $1+3+7+15+31+63+\cdots$

$= 2^1-1 + 2^2-1 + 2^3-1 + 2^4-1 + \cdots\cdots 2^{1000\,000}-1$

$= 2^1+2^2+2^3+\cdots+2^{1000\,000} - 1000\,000$

$= 2^{1000\,001} - 2 - 1000\,000$

$= 2^{1000\,001} - 1000\,002$

So $x = 1000\,001$, $y = -1000\,002$

$\therefore\ x+y = -1$

(6) Let the vector from the centre of the Earth to Poggle be $\underset{\sim}{p}$

" " " " " " " " " " " Brisbane be $\underset{\sim}{b}$

" " " " " " " " " " " Singapore be $\underset{\sim}{s}$

$$\underset{\sim}{p} = \underset{\sim}{s} \times \underset{\sim}{b}$$

Let the radius of the Earth be 1

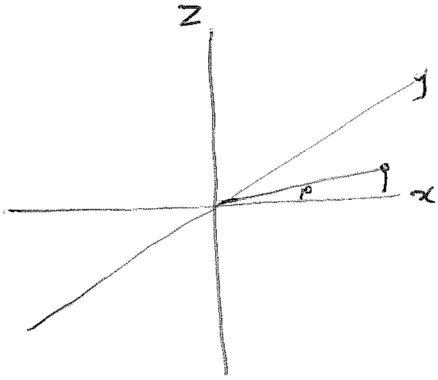
Let the zero of longitude pass through Singapore

Then, for $\underset{\sim}{s}$

$\theta = 0$

$\phi = 1^\circ$

$r = 1$



So $\underset{\sim}{s} = \cos 1^\circ \underset{\sim}{i} + \sin 1^\circ \underset{\sim}{k}$

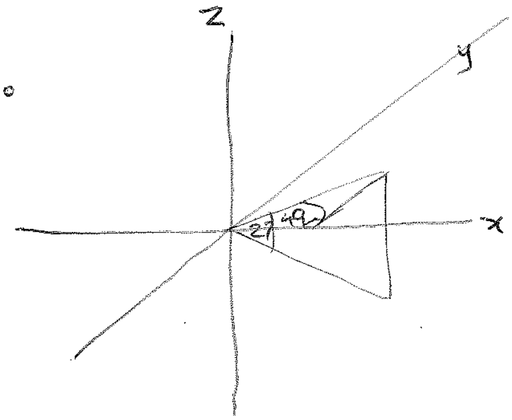
$= 0.9998 \underset{\sim}{i} + 0.0175 \underset{\sim}{k}$

For $\underset{\sim}{b}$

$\theta = 153 - 104 = 49^\circ$

$\phi = -27^\circ$

$r = 1$



So $\underset{\sim}{b} = \cos 27^\circ \cos 49^\circ \underset{\sim}{i}$

$+ \cos 27^\circ \sin 49^\circ \underset{\sim}{j}$

$- \sin 27^\circ \underset{\sim}{k}$

$= 0.5846 \underset{\sim}{i} + 0.6725 \underset{\sim}{j} - 0.4540 \underset{\sim}{k}$

$$\underset{\sim}{h} = \underset{\sim}{s} \times \underset{\sim}{b}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0.9998 & 0 & 0.0175 \\ 0.5846 & 0.6725 & -0.4540 \end{vmatrix}$$

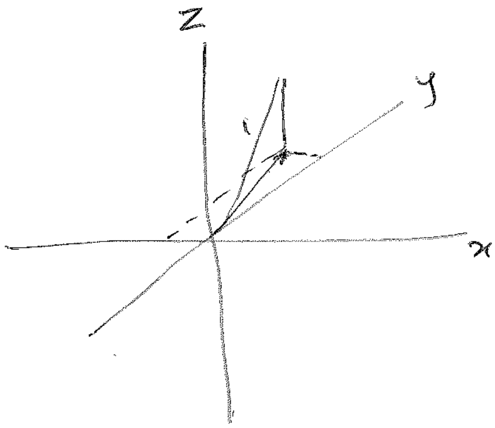
$$= -0.0118\,\underset{\sim}{i} + 0.4657\,\underset{\sim}{j} + 0.6724\,\underset{\sim}{k}$$

<u>Poggles' position</u>

$$\phi = \sin^{-1} 0.6724 = 42.4^\circ \text{ N}$$

$$\theta = \tan^{-1} \frac{0.0118}{0.4657} + 90^\circ$$

$$= 91.45$$



$\therefore$ latitude is 42°N

longitude is $104 + 91^\circ$E

$= 195^\circ$E

$= 165^\circ$W

So Poggles is at 42°N 165°W